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1982 J. Phys. A: Math. Gen. 15 2259

(http://iopscience.iop.org/0305-4470/15/7/035)

J. Phys. A: Math. Gen. **15** (1982) 2259–2266. Printed in Great Britain

Series analysis study of bond–site percolation on an anisotropic triangular lattice

G M Torrie†, D S Gaunt‡, A J Guttmann§‖ and S G Whittington§

† Department of Mathematics and Computer Science, Royal Military College, Kingston, Canada

‡ Department of Physics, King's College, University of London, Strand, London, UK

§ Department of Chemistry, University of Toronto, Toronto, Canada, M5S 1A1

Received 21 December 1981

Abstract. We derive low-density series expansions for the mean size of finite clusters on an anisotropic triangular lattice. By varying a bond density parameter this model includes site percolation on the square and triangular lattices and interpolates smoothly between the two. We identify the critical line and our results are consistent with the critical exponent γ being constant along this line.

1. Introduction

There have been a number of recent discussions of the question of universality of critical exponents in bond–site percolation, using real space renormalisation group (Nakanishi and Reynolds 1979, Napiorkowski and Hemmer 1980, Guttmann and Whittington 1982) and series analysis methods (Agrawal *et al* 1979). The renormalisation group treatments provide strong evidence for universality on the square lattice, the square lattice with next-neighbour bonds and the anisotropic triangular lattice. These renormalisation group methods are less effective at accurately locating the critical surface, but Agrawal *et al* (1979) have used series analysis methods to determine the critical line for bond–site percolation on the square lattice.

In this paper we shall describe a series analysis investigation of a bond–site percolation problem on an anisotropic triangular lattice. A triangular lattice can be distorted (see figure 1) to form a square lattice in which each face has an additional diagonal bond, so that each lattice site has coordination number six. However, we can now distinguish two classes of bonds (drawn as full and broken lines in figure 1). We shall call the horizontal and vertical bonds 'square bonds' to distinguish them collectively from the diagonal bonds. We consider the percolation problem in which square bonds are present (or active) with probability one, diagonal bonds are present with probability d and sites are present with probability s. (This is a particular case of the more general problem considered in the previous paper by Guttmann and Whittington (1982) in which square bonds were present with probability b, but that model was studied only by real space renormalisation group methods.) If $d = 0$ we have site percolation on the square lattice while $d = 1$ gives site percolation on the triangular lattice. Varying d allows us to pass from the square to the triangular lattice

‖ Permanent address: Department of Mathematics, University of Newcastle, NSW 2308, Austrailia.

0305-4470/82/072259+08$02.00

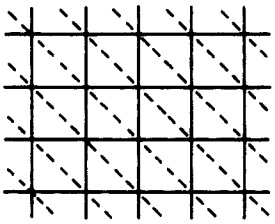

Figure 1. An anisotropic triangular lattice deformed into a square lattice with additional diagonal bonds.

problem. For a range of values of $d = 0$ (0.1) 1 we have estimated the critical site density $s_c(d)$. Plotting $s_c(d)$ against d then determines the critical line, separating the non-percolating region from the percolating region. This critical line corresponds to the intersection of the critical surface with the plane $b = 1$ in Guttmann and Whittington (1982). The location of this critical line has also been estimated by Hoshen *et al* (1979), using Monte Carlo methods. We also study the behaviour of the critical exponent γ characterising the divergence of the mean size of finite clusters along this line. Our results are consistent with γ being independent of d.

2. Series derivation

We have enumerated all bond–site clusters containing up to thirteen sites. We first use the method described by Martin (1974) to generate all clusters which are strongly embeddable in the triangular lattice. Each of these triangular lattice site clusters can be viewed as a cluster strongly embeddable in the square lattice, containing in addition zero or more diagonal bonds. For each such cluster we determine the generalised site perimeter consisting of

(i) r perimeter sites connected to the cluster by at least one square bond,

(ii) t_1 perimeter sites connected to the cluster only by exactly one diagonal bond, and

(iii) t_2 perimeter sites connected to the cluster only by exactly two diagonal bonds.

Not more than two diagonal bonds can be incident upon any one site. An example of a 10-site cluster with $r = 16$, $t_1 = 2$, $t_2 = 1$ and three diagonal bonds is shown in figure 2. To each triangular lattice site cluster with n sites and k diagonal bonds there corresponds 2^k *potential* bond–site clusters, each with the same site perimeter (r, t_1, t_2). The factor 2^k arises since each of the k diagonal bonds can be present or absent. Each of these combinations in which $\bar{h}$ diagonal bonds are removed without decomposing the cluster into two or more connected components produces a bond–site cluster with $h = k - \bar{h}$ diagonal bonds. Such a bond–site cluster appears with probability

$$p = s^n(1-s)^r d^h (1-d)^{\bar{h}} (1-sd)^{t_1} (1-2sd+sd^2)^{t_2} \tag{2.1}$$

where s and d are the (uniform) densities of sites and diagonal bonds, respectively.

The determination of which sets of diagonal bonds may be deleted without destroying the connectedness of the cluster (i.e. the sets of diagonal bonds which are not cut sets of the cluster) is simplified because those diagonal bonds appearing as a bond in an elementary triangle (see figure 1) may be independently deleted without affecting the connectivity. When all such diagonal bonds are removed, the resulting

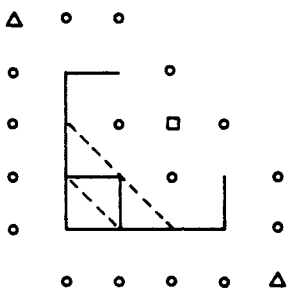

Figure 2. A ten-site cluster on the anisotropic triangular lattice with three diagonal bonds. The perimeter sites shown as circles are 'connected' to the cluster through at least one horizontal or vertical bond. The perimeter sites shown as triangles are 'connected' to the cluster only through precisely one diagonal bond and those shown as squares only through precisely two diagonal bonds.

cluster will contain from zero to three circuits of five or more edges (for clusters with less than fourteen sites). If the resulting cluster is a tree, no remaining diagonal bonds can be removed. Resulting clusters with at least one circuit and at least one diagonal bond are 'trimmed' by successively removing vertices of degree one and edges incident on such vertices. Any diagonal bonds removed in this trimming are essential and must be present in all derived bond–site clusters. The number of possible topologies of clusters which still contain a diagonal bond after trimming is sufficiently small that such clusters can be handled by special purpose subroutines, or counted by hand. Those cases treated by hand counting were checked independently by at least three of us.

If we write $c(n, h, \bar{h}, r, t_1, t_2)$ as the number of bond–site clusters (per site) with n sites, h occupied diagonal bonds between cluster sites, $\bar{h}$ unoccupied diagonal bonds between cluster sites, and generalised site perimeter (r, t_1, t_2), then the number of bond–site clusters at site density s and diagonal bond density d is given by

$$K(s, d) = \sum c(n, h, \bar{h}, r, t_1, t_2)p \tag{2.2}$$

where p is given by (2.1) and the summation is over all values of n, h, $\bar{h}$, r, t_1 and t_2. We can define a generalised perimeter polynomial $D_{n,h}$ by writing (2.2) as

$$K(s, d) = \sum_n s^n \sum_h d^h D_{n,h}(s, d). \tag{2.3}$$

This forms a useful check on the data since $D_{n,0}(s, 0)$ reduces to the perimeter polynomial for the site problem on the square lattice and $\Sigma_h D_{n,h}(s, 1)$ reduces to the perimeter polynomial for the site problem on the triangular lattice.

We can define the percolation probability $P(s, d)$ as the probability that a randomly chosen occupied site is a member of an infinite cluster, and $P(s, d)$ is then given by

$$P(s, d) = 1 - s^{-1} \sum_{n,h} n s^n d^h D_{n,h}(s, d). \tag{2.4}$$

We note that all coefficients in a low-density expansion of $P(s, d)$ (i.e. in powers of s and d) must be identically zero. This forms a further important check on the enumeration.

Table 1. Coefficients in the low-density expansion for the mean size $S(s, d) = 1 + \Sigma_{m,n} a_{mn} s^m d^n$.

m \ n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	4	2												
2	12	4	2											
3	24	18	4	2										
4	52	44	24	4	2									
5	108	120	36	30	4	2								
6	224	216	208	60	36	4	2							
7	412	626	318	262	20	42	4	2						
8	844	1 100	1 274	176	570	56	48	4	2					
9	1 528	2 994	944	3 124	−278	520	−24	54	4	2				
10	3 152	3 760	9 626	−2 068	5 786	−756	1 294	32	60	4	2			
11	5 036	15 446	−2 662	27 774	−11 618	13 156	−3 452	978	−96	66	4	2		
12	11 984	8 196	66 346	−48 732	84 908	−43 376	25 256	−3 792	2 536	−12	72	4	2	
13	15 040	79 846	−77 920	284 142	−254 252	238 686	−104 124	45 294	−11 972	1720	−196	78	4	2

Agrawal *et al* (1979) have pointed out that the mean size of the cluster can be defined in several ways. We have chosen a definition based on the number of sites in the cluster, i.e.,

$$S(s, d) = s^{-1} \sum_{n,h} n^2 s^n d^h D_{n,h}(s, d). \tag{2.5}$$

Since we have configurational data on clusters with up to and including thirteen sites, we can immediately expand (2.5) in powers of s and d, correct through order s^{12}. We have obtained an additional term, the coefficient of s^{13}, by noticing that the coefficient of s^{13} in (2.4) must be identically zero (Sykes and Glen 1976). The series coefficients are given in table 1. The first column (the coefficients of d^0) agrees with the coefficients given by Sykes and Glen (1976) for site percolation on the square lattice, and the row sums agree with the corresponding coefficients given there for the triangular lattice.

3. Analysis of series

We first consider the problem of locating the critical line which divides the non-percolating region from the percolating region in the (s, d) unit square. The critical values of s when $d = 0$ or 1 are well known from a series study of site percolation on the square lattice, $s_c(0) = 0.593$ (Sykes *et al* 1976) and from the exact result, $s_c(1) = \frac{1}{2}$, for the triangular lattice site problem (Sykes and Essam 1964).

Since s and d can be varied independently, we could approach the critical line from the low-density region in any desired direction. The experience of Agrawal *et al* (1979) for bond–site percolation on the square lattice seems to suggest that one should approach the critical line approximately perpendicularly and we have consequently chosen to vary s for a set of fixed values of $d = 0$ (0.1) 1.0. Since the line joining $s_c(0) = 0.593$ to $s_c(1) = 0.5$ will presumably be of small gradient in the d–s plane (with average absolute value <0.1), holding d fixed corresponds to a vertical approach to the critical line, which is therefore close to perpendicular. For each value of d we form a series for S in ascending powers of s and analyse each of these series using standard series analysis methods (Gaunt and Guttmann 1974), where we are making the standard assumption that near $s_c(d)$ the function behaves like

$$S(s, d) \sim B(d)(1 - s/s_c(d))^{-\gamma(d)}. \tag{3.1}$$

As a first step we formed diagonal and off-diagonal Padé approximants to the logarithmic derivative of S. In each case these approximants indicate a singularity on the positive real axis with strongly divergent behaviour. However, there is also evidence of a singularity on the negative real axis, and additional singularities in the complex plane which are closer to the origin and which must eventually dominate the higher order terms in the series. The behaviour of the Padé approximants gradually deteriorates as d decreases but, at least for $d \geq 0.4$, the estimates of the exponent associated with the physical singularity seem to be independent of d, though the location of the physical singularity decreases smoothly as d increases for all d in the range 0–1. This is in agreement with the results of a real space renormalisation group treatment (Guttmann and Whittington 1982) and, for this reason, we tentatively assume that the exponent γ is independent of d, and form estimates of the critical site density $s_c(d)$, assuming a constant value for γ. We choose the conjectured value

of $\gamma = 43/18$ (den Nijs 1979, Nienhuis *et al* 1979, Pearson 1980). If this is the correct value of γ, then $[S(s, d)]^{1/\gamma}$ should be meromorphic, with the asymptotic form

$$S(s, d)^{1/\gamma} \sim B(d)^{1/\gamma}(1 - s/s_c(d))^{-1} \tag{3.2}$$

and it should be possible to estimate $s_c(d)$ as poles of Padé approximants to $S(s, d)^{1/\gamma}$. Using this method we are able to obtain quite precise estimates of the critical points $s_c(d)$ for a variety of values of d. The critical curve derived in this way is given by table 2, in good agreement with the Monte Carlo estimates of Hoshen *et al* (1979).

Table 2. Estimates of the critical line and critical amplitudes.

d	$s_c(d)$ (Padé)	$s_c(d)$ (ratios)	$B(d)$
0	0.593	—	0.583
0.1	0.584	—	0.589
0.2	0.576	—	0.626
0.3	0.5663	0.57	0.638
0.4	0.5580	0.56	0.675
0.5	0.5485	0.55	0.693
0.6	0.5390	0.54	0.705
0.7	0.5295	0.53	0.722
0.8	0.5196	0.52	0.733
0.9	0.5097	0.51	0.743
1.0	$\frac{1}{2}$	$\frac{1}{2}$	0.757

In spite of the closer singularities both on and off the negative real axis, all terms which we have derived in the series are positive (though the distribution of singularities is such that this behaviour cannot persist), so that ratio methods may give useful estimates of the behaviour near the physical singularity. In fact we find that for $d \geqslant 0.3$ the ratios are smooth and so may be extrapolated to give estimates of $s_c(d)$. These estimates are less precise than those from the Padé approximants to (3.2) but are nevertheless completely consistent with these values. We have summarised these results for $s_c(d)$ in table 2, where we believe the estimates to be correct up to an uncertainty in the last digit quoted, subject of course to the assumption that $\gamma = 43/18$ for all d.

As a consistency check, we have used the values of $s_c(d)$ obtained above and then formed biased estimates of γ in the usual way. That is, we have formed diagonal and off-diagonal Padé approximants to $(s_c(d) - s)(\mathrm{d}/\mathrm{d}s) \ln S(s, d)$ and evaluated these at $s = s_c(d)$. Such approximants should give estimates of $\gamma(d)$. The results are consistent with a value of γ between 2.38 and 2.43 for all values of d.

We have also estimated the amplitudes $B(d)$ in (3.1) assuming that $\gamma = 43/18$ and the values of $s_c(d)$ given in table 2, by evaluating Padé approximants to $(s_c(d) - s)S(s, d)^{1/\gamma} \sim s_c(d)B(d)^{1/\gamma}$. These estimates are also given in table 2. They are smoothly increasing as d increases.

From table 2 we see that $s_c(d)$ is virtually a linear function of d, and in fact the curve lies above the straight line joining $s_c(0)$ to $s_c(1)$ by at most $\frac{1}{2}\%$. This is in precise agreement with the corresponding results for the real space renormalisation group calculation of Guttmann and Whittington (1982), for which the curve also lies above the straight line by at most $\frac{1}{2}\%$, though the values of $s_c(0)$ and $s_c(1)$ calculated there are lower than the values calculated here. The estimates of $B(0)$ and $B(1)$ given in

table 2 are some 17% higher than the corresponding estimates quoted by Sykes *et al* (1976), but their estimates, though based on the same values of s_c, used a higher value of γ (2.43) to determine B. Thus our estimates of B, like those of $s_c(d)$ are expected to be correct up to an uncertainty in the last digit quoted, subject to the assumption that $\gamma = 43/18$ for all d.

4. Discussion

We have considered an anisotropic triangular lattice in which one set of bonds (constituting a square lattice) is present with probability one. The bonds in the second set are present with probability d and, as d varies from zero to unity we pass from the square lattice to the triangular lattice. In addition, sites are present with probability s, and we then have a bond–site percolation process which, at the 'diagonal' bond densities $d = 0$ and $d = 1$, corresponds to site percolation on the square and triangular lattices. We have derived low-density series for the mean size of a finite cluster as a function of s and d, using configurational data for clusters with up to thirteen sites. Using standard series analysis methods we have estimated the location of the critical line and our results are in good agreement with the Monte Carlo work of Hoshen *et al* (1979). In addition our results suggest that the critical exponent (γ) is constant everywhere on this line. We have also estimated critical amplitudes and these vary smoothly along the critical line.

Our results therefore support the universality of the critical exponents for site percolation on the square and triangular lattices as well as in this more general bond–site percolation process, in agreement with predictions from a real space renormalisation group treatment for a more general model (Guttmann and Whittington 1982).

Apart from their interest from the point of view of universality in percolation processes, these results may also be of relevance to the problem of polymer gels. Bond–site percolation has been used as a model of this gelation process (Coniglio *et al* 1979) but the question of whether this process should be described by a percolation model or by a branching process model is contentious (Stauffer 1981). The work discussed here and in Guttmann and Whittington (1982) suggests that bond–site percolation with *two different types of bonds*, is in the same universality class as standard site and bond percolation processes.

Acknowledgments

This work was financially supported by NSERC of Canada and by NATO.

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